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ORIGINAL PAPER

The geometry of optimal partitions in location problems

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Abstract Given the position of some facilities, we study the shape of optimal partitions of the customers' area in a general planar demand region minimizing total average cost that depends on a set up cost plus some function of the travelling distances. By taking into account different norms, according to the considered situation of the location problem, we characterize optimal consumers' partitions and describe their geometry. The case of dimensional facilities is also investigated.

Keywords Facility location · Optimal transport · Polyhedral- and ℓ_p-norms

1 Introduction

Most research in location analysis focuses on locating facilities to better cover customers' demand. This traditional analysis can be seen as a primal approach. Alternatively, from a dual point of view, one can approach the location problem from a different perspective, namely the allocation of customers to given facilities. This is also an important aspect in the area of location and it is usually done by means of some allocation rules such as most preferred, cheapest or closest. Obviously, the choice of an allocation rule has an important impact in the final solution since the same set of facilities may give rise to different solutions.

The problem that we wish to consider in this paper falls within the latter area. We wish to design "optimal" districting of regions of demand originated from given set of

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facilities and using specific criteria. In our setting districting means to find a partition of an area into smaller areas with optimal properties with respect to our allocation rule. See Kalcsics [14] for a recent overview of this subject. Districting problems are motivated by many different applications: political, territorial sales, school areas, waste collection, et cetera. Among the many criteria that have been used in the literature: balancedness, contiguity, compactness, closeness,... we shall restrict ourselves to closeness. From a pure economical point of view, and assuming that distances are a proxy for costs, we are interested in most economical or efficient partitions. Therefore, our approach could be used in some applications but not in some other as for instance in political districting where some other considerations should be made. Our problem is related, but different, from the market area problem [4,17,29]. We observe that in the market area problem one is searching for the simultaneous determination of production levels and distribution patterns. However, the distribution patterns do not induce subdivisions on the space because it is allowed multisource coverage of demand. In our setting, this is not possible because the allocation rules assign each potential demand to a unique facility. This has an impact on the solution since it introduces some source of discreteness that makes the problem more challenging.

From a pure mathematical perspective, the problem can be stated as: given a set of facilities in a Borel set $\Omega \in \mathbb{R}^2$, which represent the demand, find a partition (up to negligible sets) into subsets such that the overall (average) distance covered from customers to their assigned facilities is minimized [8,9,26]. This question is related with the problem of finding the best approximation, by discrete measures, of a density function on a compact subset of $\mathbb{R}^n$ in the sense of the p-Wasserstein distance (see [5]). This connection has been only partially applied to get actual solutions in the field of Location Analysis and we will show later how to exploit it. In addition, there is another body of literature on this subject which is related to the so called Voronoi or generalized Voronoi diagrams [25]. It is well-known that Voronoi diagrams exhibit closeness properties with respect to their centers and the considered distance measure. We will extend this optimality property with respect to average distances and we shall prove a characterization of optimal shapes for partitions under general cost functions that are bivariate polynomials of the coordinates of the distances between the facilities and their assigned customers. The reader should observe that since polynomials are dense in the set of continuous functions by our approach we are solving (up to any degree of accuracy) the general partition problem with continuous functions.

The paper is organized as follows: in Sect. 2 we recall some optimal transport results and present the facility location problem; in Sect. 3 we study the optimal partition of the domain depending on the cost function form and in Sect. 4 we consider the circular facilities case. Some concluding remarks are drawn in Sect. 5.

2 Optimal transport formulation

2.1 Optimal transport results

We consider a bounded Borel subset of $\mathbb{R}^2$, say Ω and $\mathcal{P}(\Omega)$, $\mathcal{P}(\Omega \times \Omega)$ the set of Borel probability measures on Ω and $\Omega \times \Omega$ respectively. Here $\mathcal{L}^2$ stands for the

2-dimensional Lebesgue measure on $\mathbb{R}^2$, and we consider the indicator function $\mathbf{1}_A(u)$ for every Borel set $A \in \Omega$ defined by 1 if $u = (x, y) \in A$ and by 0 if $u \notin A$.

For $\mu \in \mathcal{P}(\Omega)$ and a Borel map $T : \Omega \to \Omega$ we shall denote by $T_\sharp \mu(B)$ the push forward (or image measure) of μ through T, which is defined as $T_\sharp \mu(B) = \mu(T^{-1}(B))$, for every Borel subset B of Ω, or equivalently, by the change of variable formula

$$\int_\Omega \varphi d T_\sharp \mu = \int_\Omega \varphi(T(u)) d\mu(u)$$

for any bounded Borel function $\varphi : \Omega \to \mathbb{R}$. A transport map between μ and ν, with $\mu, \nu \in \mathcal{P}(\Omega)$, is a Borel map T such that $T_\sharp \mu = \nu$. Now, let $c : \Omega \times \Omega \to [0, +\infty]$ a Borel cost function, the *Monge optimal transport* problem [22] for the cost c consists in finding a transport T between μ and ν that minimizes the total cost, i.e. solution to

$$\inf_{T_\sharp \mu = \nu} \int_\Omega c(u, T(u)) d\mu(u). \tag{M}$$

The minimizer is called an *optimal transport map*. The Monge problem is, in general, difficult to solve and it can be useful to consider the Kantorovich relaxed Monge's formulation [15] as

$$\mathcal{W}_c(\mu, \nu) = \inf_{\gamma \in \Pi(\mu, \nu)} \int_{\Omega \times \Omega} c(u, v) d\gamma(u, v), \tag{MK}$$

where $\Pi(\mu, \nu)$ denotes the set of all transport plans between μ and ν, i.e. Borel probability measures on $\Omega \times \Omega$ having μ and ν as marginals, i.e. such that $(\pi_1)_\sharp \gamma = \mu$ and $(\pi_2)_\sharp \gamma = \nu$, where we denote by π_1 the projection onto the first component and by π_2 the projection onto the second component. Since $\Pi(\mu, \nu)$ is weakly$*$ compact and c is continuous, it is easy to see that the infimum of $\mathcal{W}_c(\mu, \nu)$ is attained at some γ and such γ is called *optimal transport plan* for the cost c between μ and ν. If there exists an optimal γ_T which is induced by a transport map, i.e. of the form

$$\gamma_T = (\mathrm{id} \times T)_\sharp \mu$$

for some transport map T, then T is an optimal solution of the Monge's problem. It turns out that $\mathcal{W}_c$ is a distance on $\mathcal{P}(\Omega)$, called the Wasserstein distance, and it metrizes the weak convergence of measures. If Ω is a compact set and c is a lower semicontinuous function on $\Omega \times \Omega$, the following duality formula holds (Theorem 3.1 in [3]):

$$\mathcal{W}_c(\mu, \nu) = \sup_{\varphi(u) + \psi(v) \le c(u, v)} \int_\Omega \varphi(u) d\mu(u) + \int_\Omega \psi(v) d\nu(v) \tag{2.1}$$

where $\varphi \in L^1_\mu(\Omega)$, $\psi \in L^1_\nu(\Omega)$. Here we denote by $L^1_\mu(\Omega)$ the vector space consisting of the Borel functions which are μ-integrable.

Existence and uniqueness results for the optimal transport problem are very difficult to obtain. Recently the existence of an optimal transport map has been proved under suitable assumptions, and also its uniqueness with additional requirements (see [1,6,16,30] and the references in). Recall the following existence theorem (Theorem 2.1 in [1]):

Theorem 2.1 *Suppose that c is a lower semicontinuous function on $\Omega \times \Omega$. Then there exists $\gamma \in \mathcal{P}(\Omega \times \Omega)$ solving (MK). Moreover, if c is continuous and real valued, provided μ has no atoms, we have*

$$\mathcal{W}_c(\mu, \nu) = \inf_{T_\sharp \mu = \nu} \int_\Omega c(u, T(u)) d\mu(u).$$

Analogously to Proposition 2.4 in [10], we obtain the following results. We use in the following the notation $N = \{1, \ldots, n\}$ and $\mu - a.e.u \in \Omega$ means for almost every $u \in \Omega$.

Proposition 2.1 *Let μ be absolutely continuous with respect to $\mathcal{L}^2$ and $D : \Omega \to \mathbb{R}$ be a nonnegative function such that $\mu(u) = D(u)du$; let $p_1, \ldots, p_n$ in Ω.*
(i) Let $\nu = \sum_{i \in N} \omega_i \delta_{p_i}$ and $(A_i)_{i \in N}$ the partition of Ω such that the map $T(u) = \sum_{i \in N} p_i \mathbf{1}_{A_i}(u)$ is an optimal transport map from μ to ν. Let moreover the pair (φ, ψ) be any solution of the dual formulation (2.1). Then, for $D - a.e.u \in \Omega$, we have

$$\varphi(u) = \inf_{i \in N} \{c(u, p_i) - \psi(p_i)\} = \sum_{i \in N} (c(u, p_i) - \psi(p_i)) \mathbf{1}_{A_i}(u). \tag{2.2}$$

(ii) Let $(A_i)_{i \in N}$ be a partition of Ω and set $\omega_i = \int_{A_i} D(u)du$, $\nu = \sum_{i \in N} \omega_i \delta_{p_i}$ and $T(u) = \sum_{i \in N} p_i \mathbf{1}_{A_i}(u)$. Let moreover $\varphi \in L^1_\mu(\Omega)$, $\psi \in L^1_\nu(\Omega)$ be two functions satisfying condition (2.2), then we have that T is optimal for (M) and the pair (φ, ψ) is optimal for the dual formulation (2.1).

Note that (2.2) describes the shape of the set A_i, for every $i \in N$, of the partition:

$$A_i = \{u \in \Omega : c(u, p_i) - \psi(p_i) < c(u, p_j) - \psi(p_j) \, \forall j \neq i\} \tag{2.3}$$

It is possible to prove that the optimal transport map from an absolutely continuous measure to an atomic measure is unique $D - a.e.$ [10].

2.2 The facility location problem

We suppose that in Ω, a Borel, compact subset of $\mathbb{R}^2$, customers are distributed according to a demand density $D \in \mathcal{L}^2(\Omega)$ that is an absolutely continuous probability measure, where $D : \Omega \to \mathbb{R}$ is a nonnegative function with unit integral

$$\int_{\Omega} D(u)du = 1$$

with $u = (x, y) \in \Omega$ and $du = dxdy$, and that n facilities $p_1, \ldots, p_n$, $p_i = (x_i, y_i) \in \Omega$ for any $i \in N = \{1, 2, \ldots, n\}$ are located in the set ($n \in \mathbb{N}$). Facility p_i serves the consumer demand in the region $A_i \subseteq \Omega$ according to a partition of the set Ω, i.e. a finite family $(A_i)_{i \in N}$ of pairwise disjoint (up to D-negligible sets) Borel sets $A_i \in \Omega$ such that $\cup_{i=1}^{n} A_i = \Omega$. For any $i \in N$, the density of the sub-region A_i will be denoted by

$$\omega_i = \int_{A_i} D(u)du.$$

We denote by N the set of the facilities $N = \{1, 2, \ldots, n\}$, by $\mathcal{A}_n$ the set of all partitions in n sub-regions of the region Ω, $A = (A_1, \ldots, A_n) \in \mathcal{A}_n$ and by S the unit simplex in $\mathbb{R}^n$ defined by $S = \left\{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n : \ \omega_i \geq 0, \ \sum_{i=1}^{n} \beta_i = 1\right\}$. Clearly, $(\omega_1, \ldots, \omega_n) \in S$.

For any $i \in N$, we assume the standard cost structure associated to costumers in the region $A_i \subseteq \Omega$ given by

$$a_i\omega_i + \int_{A_i} F_i(u - p_i)^{r_i} D(u)du$$

where $a_i \in [0, +\infty[$ is the fixed set up cost of the facility p_i and the second term represents the service costs of customers in A_i given by a general bivariate polynomial $F_i \in \mathbb{R}[XY]$ and $r_i \in \mathbb{Q}_+$. In many cases, as it is usual in location analysis, the dependence on the pair (u, p_i) is given by a measure of the distance from p_i to u, namely $\gamma_Q(u - p_i)$ where γ_Q is the Minkowski functional of Q, a compact, convex set with the origin in its interior. The reader may observe that the choice of $F_i \in \mathbb{R}[XY]$ ensures that we can handle any continuous cost function since the set of polynomials is dense in the space of continuous functions. On the other hand, by the dependence through γ_Q we cover all the family of polyhedral or block norms [31] and also the ℓ_q norms in $\mathbb{R}^2$, i.e.

$$\ell_q(p_i - p_j) = \left(|x_i - x_j|^q + |y_i - y_j|^q\right)^{\frac{1}{q}},$$

for $q \geq 1$ (special cases include the rectilinear or Manhattan metric when $q = 1$, the Euclidean metric when $q = 2$) and if $q = +\infty$ the Tchebycheff metric

$$\ell_\infty(p_i - p_j) = \max\{|x_i - x_j|, |y_i - y_j|\},$$

with $p_i, p_j \in \Omega$.

We will consider in the following also the case $F_i(\ell_q(u - p_i))^{r_i} = \alpha_i \ell_q(u - p_i)$ with $r_i = 1/q$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for all $i \in N$. In this case we deal with a weighted ℓ_q metric.

Given the set of the facilities $\{p_1, \ldots, p_n\}$, we are interested in the problem of finding the optimal partition of the customers minimizing the total cost

$$f(A) = \sum_{i=1}^{n} \left\{ \int_{A_i} [a_i + F_i(u - p_i)^{r_i}] D(u) du \right\}$$

and the optimization problem is

$$\min_{(A_1,\ldots,A_n)\in\mathcal{A}_n} \sum_{i\in N} \left\{ \int_{A_i} [a_i + F_i(u - p_i)^{r_i}] D(u) du \right\} \qquad (P)$$

This problem has been studied in the case $F_i(u - p_i)^{r_i} = \ell_q(u - p_i)$ in [2] and in the case where $F_i(u - p_i)^{r_i}$ is a continuous function of $\ell_2(u - p_i)$ in [28]. By using equality (2.1) and Theorem 2.1, if $\mu(u) = D(u)du$ and $c(u, p_i) = F_i(u - p_i)^{r_i}$, we have that

$$\begin{aligned} &\inf_{(A_1,\ldots,A_n)\in\mathcal{A}_n} \sum_{i\in N} \left\{ \int_{A_i} [a_i + F_i(u - p_i)^{r_i}] D(u) du, \right\} \\ &= \inf_{(\omega_1,\ldots,\omega_n)\in S} \left\{ \mathcal{W}_c(\mu, \sum_{i=1}^{n} \omega_i \delta_{p_i}) + \sum_{i=1}^{n} \omega_i a_i, \quad \omega_i = \int_{A_i} D(u) du \right\} \end{aligned} \qquad (2.4)$$

In fact, by Proposition (2.1)

$$\begin{aligned} &\inf_{(\omega_1,\ldots,\omega_n)\in S} \left\{ \mathcal{W}_c(\mu, \sum_{i=1}^{n} \omega_i \delta_{p_i}) + \sum_{i=1}^{n} \omega_i a_i, \quad \omega_i = \int_{A_i} D(u) du \right\} \\ &= \inf_{(\omega_1,\ldots,\omega_n)\in S} \int_{\Omega} F_i(u - T(u))^{r_i} D(u) du + \sum_{i=1}^{n} \omega_i a_i, \quad \omega_i = \int_{A_i} D(u) du \Big\} \\ &= \inf_{(\omega_1,\ldots,\omega_n)\in S} \sum_{i\in N} \int_{A_i} F_i(u - p_i)^{r_i} D(u) du + \sum_{i=1}^{n} \omega_i a_i, \ \ \omega_i = \int_{A_i} D(u) du \Big\} \\ &= \inf_{(A_1,\ldots,A_n)\in\mathcal{A}_n} f(A). \end{aligned} \qquad (2.5)$$

Then, summarising and adapting the proof of Lemma 2 in [20], the following existence theorem holds.

Theorem 2.2 *Suppose that F_i is a continuous function for any $i \in N$. Then the problem (P) admits a solution that verifies*

$$A_i = \Big\{ u \in \Omega : a_i + F_i(u - p_i)^{r_i} < a_j + F_j(u - p_j)^{r_j} \quad \forall j \neq i \Big\}, \qquad (2.6)$$

where the equalities is intended up to D-negligible sets.

3 Optimal partitions of the domain

Depending on the choice of the cost function c, the optimal partition of the customers has some properties that we want to investigate. We suppose in this section that F_i is a continuous function, so that result (2.2) holds. The case of $F_i(t) = t^2$ and $q = 2$ has been studied in [19,20]: given the location of the facilities, the optimal partition is done by means of polygons in the set Ω.

Let us introduce some classes of geometrical objects in the plane. We denote by Γ_P the set of the polygons of the plane.

Definition 3.1 The set of geometrical figures in the plane whose boundary is done by

1. segments and arcs of circles is denoted by Γ_O
2. segments and algebraic curves (defined by a polynomial equation in x and y) is denoted by Γ_C

Recall that an algebraic curve is any curve that can be described as the solution of a set of polynomial equations and a semialgebraic set is the one described by a system of polynomial inequalities.

We prove in the following theorem the possible configurations of the splitting of the customers: each service region A_i, i.e. the set of customers that will be served by facility p_i, has a boundary that can be a set of Γ_P, Γ_O or Γ_C depending on the chosen metric. The resulting partition of the domain is described in the following theorems.

Theorem 3.1 *Let us denote by $p_i = (x_i, y_i)$ the ith facility. Let us consider that the cost function of problem (P) is given by one of the following cases:*

1. $F_i(u - p_i)^{r_i} = \left(\sum_{h,k:h+k\leq m} b_i^{hk}|x - x_i|^h|y - y_i|^k\right)^{r_i}$ *with $r_i \in \mathbb{Z}_+$ for all $i = 1, \ldots, n$;*
2. $F_i(u - p_i)^{r_i} = \left(\beta_i + \sum_{h,k:h+k\leq m} b_i^{hk}|x - x_i|^h|y - y_i|^k\right)^{r_i}$ *with $\beta_i \in \mathbb{R}$, $r_i \in \mathbb{Q}_+$ and $a_i = a$ for all $i = 1, \ldots, n$;*

where we denote $u = (x, y)$. The optimal partition $A_1, \ldots, A_n$ solution of the problem is described by semi-algebraic sets with boundaries in Γ_C.

Proof To prove item (2), apply the characterization of optimal partitions given by Theorem 2.2 to deduce that the set of points that account for the same cost with respect to facilities i and j is described by:

$$\left(\beta_i + \sum_{h,k:h+k\leq m} b_i^{hk}|x - x_i|^h|y - y_i|^k\right)^{r_i} = \left(\beta_j + \sum_{h,k:h+k\leq m} b_j^{hk}|x - x_j|^h|y - y_j|^k\right)^{r_j}.$$

It is clear that after some algebra this equation becomes a polynomial in x, y and therefore the result follows.

The proof of item (1) is similar. □

We note in passing that the above result includes as particular cases those situations where the dependence with respect to the vectors $u - p_i$ is measured by weighted ℓ_q norms, $q \in (1, +\infty)$.

Corollary 3.1 *The optimal partition $A_1, \ldots, A_n$ solution of the problem (P) is done by sets in Γ_P, i.e. polygons, in the set Ω if:*

1. $F_i(u - p_i) = \alpha_i(|x - x_i| + |y - y_i|)$ *(Manhattan metric),* $r_i = 1$*, and* $\alpha_i > 0$ *for any* $i \in N$*;*
2. $F_i(u - p_i) = (x - x_i)^2 + (y - y_i)^2$ *(unweighted squared Euclidean metric),* $r_i = 1$ *for any* $i \in N$*;*
3. $F_i(u - p_i) = \alpha_i \max\{|x - x_i|, |y - y_i|\}$ *(Tchebycheff metric),* $r_i = 1$*, and* $\alpha_i > 0$ *for any* $i \in N$*.*

Proof Case 1. The ith facilities is given by $p_i = (x_i, y_i)$ and we denote $u = (x, y)$; by using inequality (2.6) we have under assumption 1 that $u \in A_i$ if

$$a_i + \alpha_i \, [|x - x_i| + |y - y_i|] < a_j + \alpha_j \left[|x - x_j| + |y - y_j|\right] \tag{3.1}$$

for all $j \neq i$; the set of the customers indifferent in choosing the facility in p_i and the facility p_j is given by $u \in \Omega$ such that

$$\alpha_i |y - y_i| - \alpha_j |y - y_j| = a_j - a_i + \alpha_j |x - x_j| - \alpha_i |x - x_i| \tag{3.2}$$

is a segment in Ω, so that the sets A_i are polygons given by the intersections of $n - 1$ half-planes with the set Ω.

Case 2. This case has been investigated in [19,20]. By using inequality (2.6) we have under assumption 2 that $u \in A_i$ if

$$a_i + \left[|x - x_i|^2 + |y - y_i|^2\right] < a_j + \left[|x - x_j|^2 + |y - y_j|^2\right] \tag{3.3}$$

for all $j \neq i$, that is equivalent to

$$y \left(2y_j - 2y_i\right) < x \left(2x_i - 2x_j\right) + \left(a_j - a_i + x_j^2 - x_i^2 + y_j^2 - y_i^2\right) \tag{3.4}$$

and again the sets A_i are polygons in Ω.

Case 3. By using inequality (2.6) we have under assumption 1 that $u \in A_i$ if

$$a_i + \alpha_i \max\left\{|x - x_i|, |y - y_i|]\right\} < a_j + \alpha_j \max\left\{|x - x_j|, |y - y_j|\right\} \tag{3.5}$$

for all $j \neq i$; in any of the possible four cases the boundary of the set is a segment, in two of these cases it is parallel to one coordinate axis. □

Corollary 3.2 *The optimal partition $A_1, \ldots, A_n$ solution of the problem (P) is done by sets in Γ_O if $F_i(u - p_i) = a_i + \alpha_i((x - x_i)^2 + (y - y_i)^2)$ (weighted squared Euclidean metric), $r_i = 1$, and $\alpha_i > 0$ for any $i \in N$.*

Proof By using inequality 2.6 we have that $u \in A_i$ if

$$a_i + \alpha_i \left[|x - x_i|^2 + |y - y_i|^2\right] < a_j + \alpha_j \left[|x - x_j|^2 + |y - y_j|^2\right] \tag{3.6}$$

for all $j \neq i$, that is equivalent to

$$\left(\alpha_i - \alpha_j\right) x^2 + \left(\alpha_i - \alpha_j\right) y^2 + \left(2\alpha_j x_j - 2\alpha_i x_i\right) x + \left(2\alpha_j y_j - 2\alpha_i y_i\right) y + \eta < 0 \tag{3.7}$$

being the constant $\eta = a_i - a_j + \alpha_i(x_i^2 + y_i^2) - \alpha_j(x_j^2 + y_j^2)$; the set is a circle in the plane and A_i is the intersection of $n-1$ circles and Ω. In the case where $\alpha_i = \alpha_j$ the boundary between A_i and A_j is a segment. Note that if the set A_i is not contained in Ω, then it must be intersected with Ω and its boundary contains segments of the boundary of Ω. □

By using a numerical procedure based on a genetic algorithm used in [19], we compute in the following examples the optimal partition given the facilities location. In the algorithm, the set Ω is discretized and for any facility $i \in N$ the area of the subregion A_i at step k is

$$A_i^k = \delta \sum_{l=1}^{L} \sum_{m=1}^{M} H_i^k(l,m)$$

where, by the characterization (2.6),

$$H_i^k(l,m) = \begin{Bmatrix} 1 & \text{if } a_i + F_i(u(l,m) - p_i)^{r_i} < a_j + F_j(u(l,m) - p_j)^{r_j} \ \forall j \neq i \\ 0 & otherwise \end{Bmatrix}$$

δ is a constant and $u(l,m)$ the element in the lth row and mth column of the chosen grid.

The computational complexity of the procedure that we follow is affordable. In each iteration k we consider a grid with $M_k, \times N_k$ points in $[0,1] \times [0,1]$. Next, we need to compute H_i^k at each point in the grid for each facility $i = 1, \ldots, n$. Observe that this evaluation involves n comparisons (according to the formula above). Therefore, the overall complexity of each iteration is $O(M_k N_k n^2)$. The number of iterations depends on the precision required in the final description of the partitions. We have observed that in most cases an order of several hundreds of subdivisions per coordinate axis is enough to have meaningful solutions. The conclusion is that this procedure is applicable for medium size location problems in the plane. Next, we illustrate some examples to provide some intuition about the geometry of the optimal partitions. In all cases, we consider the region $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$.

Example 3.1 Let us consider a Gaussian distribution

$$D(u) = \exp\left(-16(x-0.5)^2 - 16(y-0.5)^2\right)$$

for any $u = (x, y) \in [0,1]^2$. Four facilities are located in $p_1 = (0.25, 0.25)$, $p_2 = (0.75, 0.25)$, $p_3 = (0.75, 0.75)$, $p_4 = (0.25, 0.75)$ and the fixed costs are $a_1 = a_2 = a_3 = 1$, $a_4 = 1.3$. In this case the unweighted squared Euclidean metric is used and the resulting partition is shown in Fig. 1.

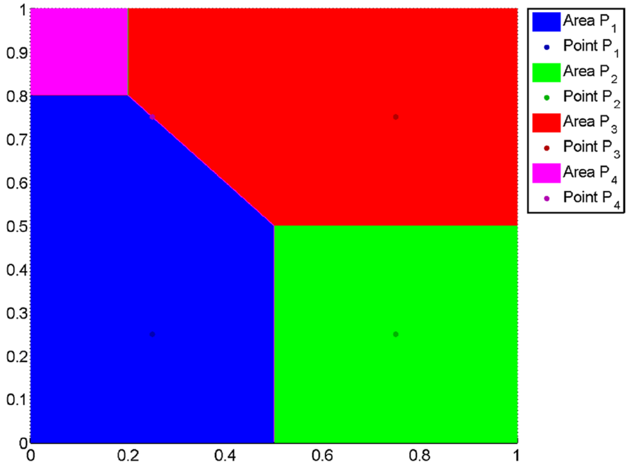

Fig. 1 Unweighted squared Euclidean metric. Example 3.1

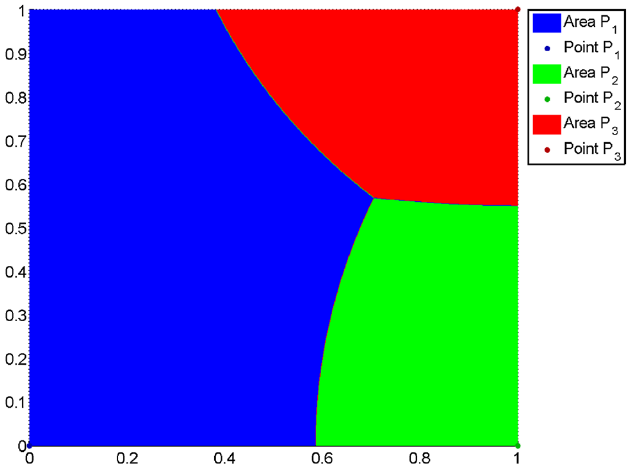

Fig. 2 Weighted squared Euclidean metric. Example 3.2

Example 3.2 Let us consider the uniform density $D(u) = 1, \forall u \in \Omega$. Three facilities are located in $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (1, 1)$ and the fixed costs are $a_1 = a_2 = a_3 = 0$, the weights $\alpha_i = i, i = 1, 2, 3$. In this case the weighted squared Euclidean metric is used and the results are summarized in Fig. 2.
Here the boundaries of the subset A_1, A_2, A_3 of the partition in the interior of Ω are arcs of the circles $(x-1)^2+(y-3)^2 = 6$ between A_3 and A_2, $(x-3/2)^2+(y-3/2)^2 = 3/2$ between A_1 and A_3 and $(x - 2)^2 + y^2 = 2$ between A_1 and A_2.

Example 3.3 Let us consider the uniform density $D(u) = 1, \forall u \in \Omega$. Three facilities are located in $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (1, 1)$ and the fixed costs are $a_1 = a_2 = a_3 = 0$, the weights $\alpha_i = i, i = 1, 2, 3$. In this case the weighted Manhattan metric is used and the resulting partition is shown in Fig. 3.

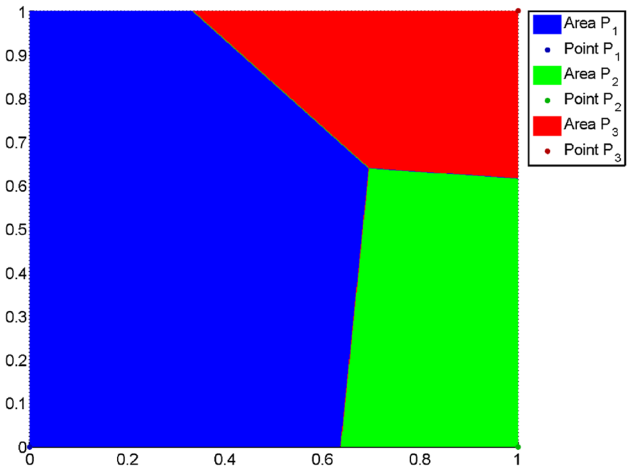

Fig. 3 Weighted Manhattan metric. Example 3.3

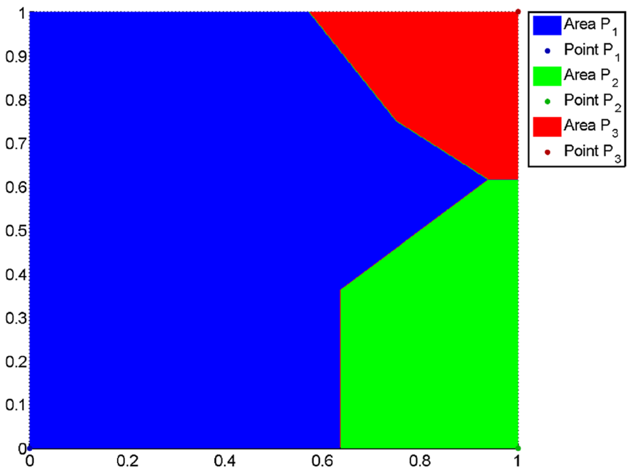

Fig. 4 Weighted Tchebycheff metric. Example 3.4

Example 3.4 Let us consider the uniform density $D(u) = 1, \forall u \in \Omega$. Three facilities are located in $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (1, 1)$ and the fixed costs are $a_1 = a_2 = a_3 = 0$, the weights $\alpha_i = i, i = 1, 2, 3$. In this case the weighted Tchebycheff metric is used. The induced partition is shown in Fig. 4.

Example 3.5 Let us consider the uniform density $D(u) = 1, \forall u \in \Omega$. Three facilities are located in $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (1, 1)$ and the fixed costs are $a_1 = a_2 = a_3 = 0$, the weights $\alpha_i = i, i = 1, 2, 3$. In this case the weighted ℓ_5 metric is used and the numerical results are summarized in Fig. 5.

It is well-known, in the specialized literature of Location Analysis, that the choice of the adequate metric plays an important role in the properties of the final solution. The

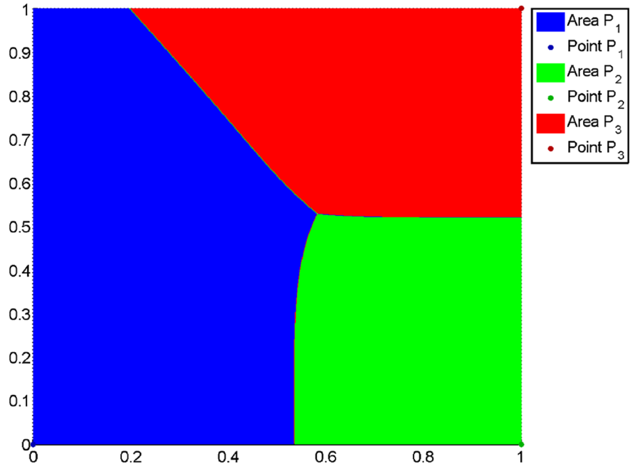

Fig. 5 Weighted ℓ_5 metric. Example 3.5

use of ℓ_p metrics is based on the properties of the induced normed spaces (including the Euclidean case). The use of squared Euclidean measurements is related to the standard theory of errors (sum of squared residuals). Finally, the use of polyhedral or block norms (which also include the ℓ_1 and ℓ_∞ norms) are often used to model real world situations (like measuring highway distances) more accurately than the standard Euclidean norm. In addition, they can also be used to approximate arbitrary norms since the set of block norms is dense in the set of all norms [18,31].

4 Dimensional facilities

In this section we consider the case where the facilities located in the domain Ω are circles in the region, as it happens in several concrete situations, for example when they are parks, cross-docking areas, etc [7,11,12,21,24,27]. We point out that for circular facilities, we cannot apply the optimal transport theory as done in Sect. 3, because the characterization (2.6) holds when the measure ν has a discrete support, and this is not the case. So, we follow a different approach that requires to obtain explicit expression of the distance from a point of the domain Ω to a circle.

We consider $P_1, \ldots, P_n$, where P_i is a circle with center $p_i = (x_i, y_i)$ and radius ϵ_i with $p_i \in \Omega$ and $\epsilon_i > 0$. We assume that any P_i is contained in the interior of Ω for any $i \in N$ and $P_i \cap P_j = \emptyset$ for any $i, j \in N, i \neq j$.

For any $u \in \Omega \backslash X$ and any closed subset $X \subset \Omega$, the distance from a point u to a set X is given by

$$d(u, X) = \min_{\xi \in X} d(u, \xi).$$

It is well-known that, for any $u \notin P_i$, in the case of Euclidean distance (Fig. 6)

$$d_2(u, P_i) = \ell_2(u - p_i) - \epsilon_i,$$

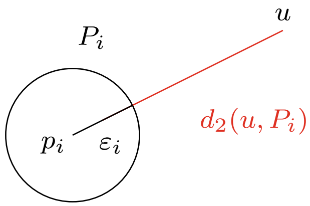

Fig. 6 Circular facilities in Ω

Moreover, as shown by Property 1 in [7], this formula holds for every ℓ_q-norm. Then we have

$$d_q(u, P_i) = \ell_q(u - p_i) - \epsilon_i.$$

for every $q \geq 1$ and $u \notin P_i$; in our discussion we always suppose that the point u cannot be taken inside the circle and we extend the formula as

$$d_q(u, P_i) = \begin{cases} \ell_q(u - p_i) - \epsilon_i & \text{if } u \notin P_i \\ 0 \ \text{ if } u \in P_i. \end{cases}$$

So that we can formulate the problem in this extensive facilities case.

Given the set of the facilities $\{P_1, \ldots, P_n\}$ and their respective set up costs $\{a_1, \ldots, a_n\}$, we are interested in finding the optimal partition of the customers minimizing the total cost

$$f(A) = \sum_{i=1}^{n} \left\{ \int_{A_i} [a_i + G_i(d_q(u, P_i))] D(u) du \right\}$$

being $G_i \in \mathbb{R}[X] : [0, +\infty[\to [0, +\infty[$ a real polynomial and the optimization problem is

$$\min_{(A_1,\ldots,A_n)\in\mathcal{A}_n^c} \sum_{i\in N} \left\{ \int_{A_i} [a_i + G_i(d_q(u, P_i))] D(u) du \right\} \qquad (PC^c)$$

where $\mathcal{A}_n^c$ is the set of all possible partitions of the set $\Omega \backslash \{P_1 \cup \cdots \cup P_n\}$.

We can consider the following relaxed formulation: finding a partition $(\bar{A}_i)_{i\in N}$ of Ω that is an optimal solution of

$$\min_{(A_1,\ldots,A_n)\in\mathcal{A}_n} \sum_{i\in N} \left\{ \int_{A_i} [a_i + G_i(d_q(u, P_i))] D(u) du \right\}. \qquad (PC)$$

Let us observe that result (2.2) is satisfied with

$$F_i(u - p_i)^{r_i} = G_i(d_q(u, P_i)) = \begin{cases} G_i(\ell_q(u - p_i) - \epsilon_i) & \text{if } u \notin P_i \\ G_i(0) \ \text{ if } u \in P_i. \end{cases}$$

and for any $i \in N$ a solution of (PC) is given by

$$\bar{A}_i = \{u \in \Omega : a_i + G_i(d_q(u, P_i)) < a_j + G_j(d_q(u, P_j))\ \forall j \neq i\}$$

Proposition 4.1 *Suppose that for any $i \in N$ we have $a_i + G_i(0) < a_j + G_j(\ell_q(u - p_j) - \varepsilon_j)\ \forall j \neq i$ and for any $u \in P_i$. Let $(\bar{A}_i)_{i\in N}$ be an optimal partition solution of the problem (PC). Then the partition*

$$(\bar{A}_i^c)_{i\in N} = (\bar{A}_i \backslash P_i)_{i\in N}$$

is an optimal solution of the problem (PC^c).

Proof For any $i \in N$ under the assumptions of non-overlapping circles, we have $P_i \subseteq \bar{A}_i$: if $u \in P_i$ since

$$\bar{A}_i = \left\{u \in \Omega : a_i + G_i\left(d_q(u, P_i)\right) < a_j + G_j\left(d_q\left(u, P_j\right)\right)\ \forall j \neq i\right\}$$

and $d_q(u, P_i) = 0$, $d_q(u, P_j) = \ell_q(u - p_j) - \varepsilon_j$ then $u \in \bar{A}_i$. Moreover for every $i \in N$

$$\int_{\bar{A}_i^c} \left[a_i + G_i\left(d_q\left(u, P_i\right)\right)\right] D(u)du = \int_{\bar{A}_i} \left[a_i + G_i\left(d_q(u, P_i)\right)\right] D(u)du.$$

□

Then the problem reduces to finding the optimal partition with respect to the set of the centers of the facilities $\{p_1, \ldots, p_n\}$ and we obtain results similar, to those already proved for the pointwise facility case, for the shape of the partition.

Remark 4.1 In the case considered in Proposition (4.1) of zero fixed costs $a_i = 0$ and $G_i(t) = t$ for any $i \in N$, the customers that are indifferent in choosing facility P_i and facility P_j are located on the so called bisector of the compact convex sets P_i and P_j [13,23,26].

Theorem 4.1 *If $G_i(t) = \alpha_i t,\ \forall i \in N,\ \alpha_i > 0$ and for any $i \in N$ we have $a_i < a_j + \alpha_j(d_q(u, p_j) - \varepsilon_j)\ \forall j \neq i$ and for any $u \in P_i$, then the external boundary of each set $\bar{A}_i^c$ of the optimal partition $\bar{A}_1^c, \ldots, \bar{A}_n^c$ solution of the problem (PC^c) is conformed by the intersection with $\Omega \backslash \bigcup_{i\in N} P_i$ of:*

1. *Polyhedra if $q = 1,\ +\infty$ or the distance is induced by the Minkowski functional of a compact, convex polyhedron with zero in its interior (respectively weighted Manhattan, weighted Tchebycheff of polyhedral metric),*
2. *Sets with boundaries defined by conics if the distance measure is the weighted squared Euclidean metric,*
3. *Semi-algebraic sets if the distance measure is induced by the q-power ℓ_q norm, $1 < q < +\infty$.*

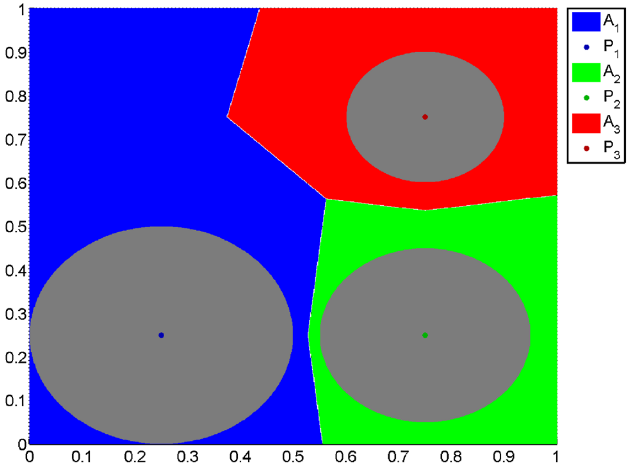

Fig. 7 Weighted Manhattan metric. Example 4.1

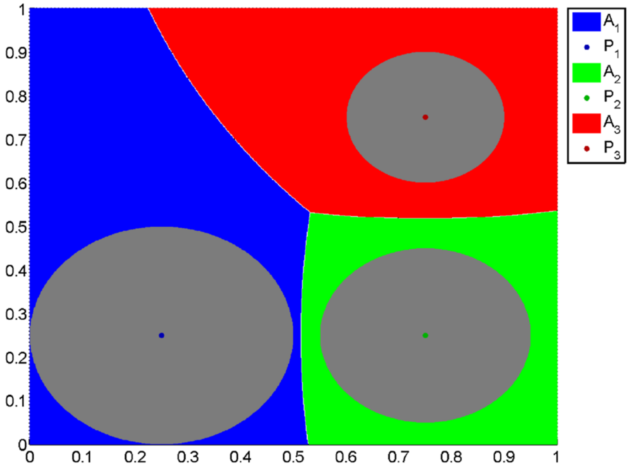

Fig. 8 Weighted Euclidean metric. Example 4.2

Proof The set $\bar{A}_i^c$ is described as

$$\bar{A}_i^c = \left\{ u \in \Omega \setminus \{P_1 \cup \cdots \cup P_n\} : a_i + \alpha_i \left(d_q(u, p_i) - \varepsilon_i \right) \right. \\ \left. < a_j + \alpha_j \left(d_q(u, p_j) - \varepsilon_j \right) \; \forall j \neq i \right\}$$

If the ith facilities is given by $p_i = (x_i, y_i)$, the inequality in the above expression results depending of the value of q: 1) for $q = 1$ $a_i + \alpha_i[|x - x_i| + |y - y_i|] < a_j + \alpha_j[|x - x_j| + |y - y_j|]$, 2) for $q \in]1, +\infty[$ is equivalent to $a_i + \alpha_i^q[|x - x_i|^q + |y - y_i|^q] < a_j + \alpha_j^q[|x - x_j|^q + |y - y_j|^q]$; and 3) for $q = +\infty$ is $a_i + \alpha_i \max\{|x - x_i|, |y - y_i|]\} < a_j + \alpha_j \max\{|x - x_j|, |y - y_j|\}$. The results follow from the above expressions. □

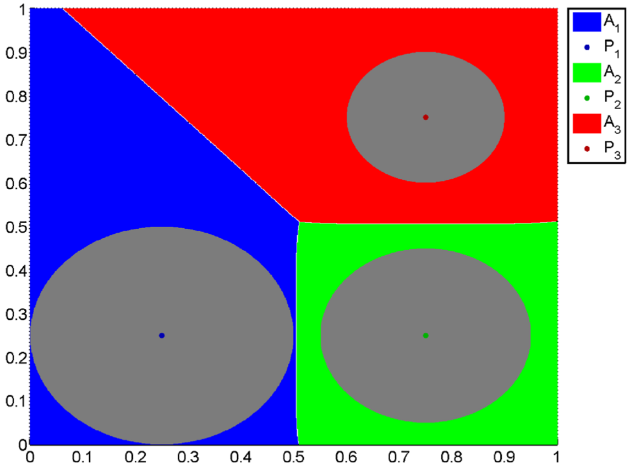

Fig. 9 Weighted ℓ_6 metric. Example 4.3

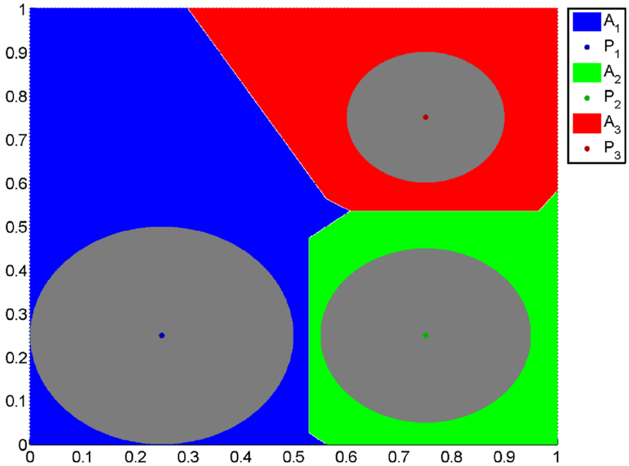

Fig. 10 Weighted Tchebycheff metric. Example 4.4

In the following we illustrate the application of the above results with several examples. In all cases, we consider the region $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, the uniform density $D(u) = 1$, three circular facilities P_1 centered in $p_1 = (0.25, 0.25)$ with radius $\varepsilon_1 = 0.25$, P_2 centered in $p_2 = (0.75, 0.25)$ with radius $\varepsilon_2 = 0.2$ and P_3 centered in $p_3 = (0.75, 0.75)$ with radius $\varepsilon_3 = 0.15$, having identical fixed costs $a_1 = a_2 = a_3 = 2$.

Example 4.1 We consider the location problem with weighted Manhattan metric with weights $\alpha_1 = 8$, $\alpha_2 = 10$ and $\alpha_3 = 13.3$. Here all the assumption of theorem (4.1) are satisfied and numerical results are summarized in Fig. 7.

Example 4.2 We consider the location problem with weighted Euclidean metric with weights $\alpha_1 = 8$, $\alpha_2 = 10$ and $\alpha_3 = 13.3$. Here all the assumption of theorem (4.1) are satisfied and numerical results are summarized in Fig. 8.

Example 4.3 We consider the location problem with weighted ℓ_6 metric with weights $\alpha_1 = 8, \alpha_2 = 10$ and $\alpha_3 = 13.3$. Here all the assumption of theorem (4.1) are satisfied and numerical results are summarized in Fig. 9.

Example 4.4 We consider the location problem with weighted Tchebycheff metric with weights $\alpha_1 = 8$, $\alpha_2 = 10$ and $\alpha_3 = 13.3$. Here all the assumption of theorem (4.1) are satisfied and numerical results are summarized in Fig. 10.

5 Conclusion

Optimal districting of regions with respect to a given set of facilities is an interesting dual location problem. We have characterized the shapes of optimal partitions in the plane with respect to general functions of the average distances. Depending on the globalizing function different structural properties of the elements in the partition are derived. In general, for the family of polyhedral (block) norms or ℓ_q-norms and bivariate polynomials we can prove that partitions are defined by semi-algebraic sets. In some particular important cases polyhedral sets are obtained, as for instance for polyhedral or the ℓ_1-, ℓ_2- and ℓ_∞-norms. We have also investigated the case of dimensional facilities and for circular shapes we obtain similar results to those mentioned above. It is an interesting open problem to determine optimal partitions for more general shapes of the dimensional facilities. Moreover, it is an interesting question to consider discontinuities in the cost functions and to analyze its implication in the consumers distribution. This can lead to a market situation where the demand is not totally satisfied. This problem opens new avenues for future research.

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